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J. Math. Anal. Appl. 300 (2004) 30–42

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Existence results to elliptic systems with nonstandard growth conditions

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Received 3 September 2003

Available online 30 September 2004

Submitted by R.H. Torres

Abstract

Nonstandard growth conditions in partial differential equations have been the subject of recent developments in elastic mechanics and electrorheological fluid dynamics [Lecture Notes in Mathematics, vol. 1748, 2000; C. R. Acad. Sci. Paris Sér. I Math. 329 (1999) 393–398; Math. USSR Izv. 29 (1987) 33–66]. In this work, elliptic systems with nonstandard growth conditions are studied. Existence and multiplicity results, under growth conditions on the reaction terms, are established.

Keywords: Nonlinear elliptic systems; Nonstandard growth conditions; Generalized Lebesgue–Sobolev spaces; Mountain pass geometry

1. Introduction

Differential and partial differential equations with nonstandard growth conditions have received specific attention in recent decades. The interest played by such growth conditions in elastic mechanics and electrorheological fluid dynamics has been highlighted in many physical and mathematical works. We can refer the reader to M. Ruzicka [15,16], V. Zhikov [19] and references therein.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.05.041

The present paper deals with existence results to elliptic systems of gradient type with nonstandard growth conditions. More precisely, we consider the following system:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \frac{\partial F}{\partial u}(x,u,v) & \text{in } \Omega, \\ -\operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) = \frac{\partial F}{\partial v}(x,u,v) & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $N \geqslant 2$, $(p,q) \in C(\bar{\Omega})^2$, $p(x) > 1$, $q(x) > 1$, for every $x \in \Omega$. The function F is assumed to be continuous in $x \in \bar{\Omega}$ and of class C^1 in $u, v \in \mathbb{R}$.

Elliptic systems with standard growth conditions have been the subject of a sizeable literature. We refer to the excellent survey article of D.G. De Figueiredo [2].

Here, problem (1) is stated in the framework of the generalized Sobolev space $W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ which will be briefly described in the following section.

2. Preliminary results and notations

For the reader's convenience, we recall some background facts concerning the generalized Lebesgue–Sobolev spaces and introduce some notations used below. We can refer the reader to the book of J. Musielak [14] and the papers of X.L. Fan et al. [4–10], P. Marcellini [11,12] and D. Edmunds and J. Rákosník [3].

Let

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}):\ h(x) > 1 \text{ for every } x \in \bar{\Omega}\},$$
$$h^+ = \max\{h(x),\ x \in \bar{\Omega}\}, \quad h^- = \min\{h(x),\ x \in \bar{\Omega}\} \quad \text{for every } h \in C_+(\bar{\Omega}),$$

and introduce, for $p \in C_+(\bar{\Omega})$, the space

$$L^{p(x)}(\Omega) = \left\{ u \text{ measurable real-valued function in } \Omega:\ \int_\Omega |u(x)|^{p(x)}\,dx < +\infty \right\}.$$

Endowed with the norm

$$|u|_{L^{p(x)}(\Omega)} = \inf\left\{ \lambda > 0:\ \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leqslant 1 \right\},$$

the space $L^{p(x)}(\Omega)$ possesses the Banach structure. The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$, $\forall x \in \Omega$.

As in the standard case, the space $W^{1,p(x)}(\Omega)$ is defined as follows:

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}:\ |\nabla u| \in L^{p(x)}\}.$$

Endowed with its natural norm

$$\|u\|_{W^{1,p(x)}(\Omega)} := |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)},$$

$W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space. If we denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, then the Poincaré inequality

$$|u|_{L^{p(x)}(\Omega)} \leqslant C|\nabla u|_{L^{p(x)}(\Omega)}$$

holds true. Moreover, $W_0^{1,p(x)}(\Omega)$, endowed with the norm $|\nabla u|_{L^{p(x)}(\Omega)}$, is a separable and reflexive Banach space. Notice that the inequality

$$\int_\Omega |u|^{p(x)}\,dx \leqslant C\int_\Omega |\nabla u|^{p(x)}\,dx$$

is false in general [9]. Let us define, for every $x \in \Omega$,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geqslant N. \end{cases}$$

As in the classical case, the embedding $W_0^{1,p(x)}(\Omega) \subset L^{r(x)}(\Omega)$ is continuous (respectively compact) if $r(x) \leqslant p^*(x)$ (respectively $r(x) < p^*(x)$), for all $x \in \bar{\Omega}$. For the other properties satisfied by the generalized Lebesgue and Sobolev spaces, we refer the reader to [9]. However, we recall the following

Lemma 1 (see [9]). *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx, \quad \forall u \in L^{p(x)}(\Omega),$$

then

(i) $|u|_{L^{p(x)}(\Omega)} < 1$ (*respectively* $= 1; > 1$) $\Leftrightarrow \rho(u) < 1$ (*respectively* $= 1; > 1$);
(ii) $|u|_{L^{p(x)}(\Omega)} > 1 \Rightarrow |u|^{p^-}_{L^{p(x)}(\Omega)} \leqslant \rho(u) \leqslant |u|^{p^+}_{L^{p(x)}(\Omega)}$;
(iii) $|u|_{L^{p(x)}(\Omega)} < 1 \Rightarrow |u|^{p^+}_{L^{p(x)}(\Omega)} \leqslant \rho(u) \leqslant |u|^{p^-}_{L^{p(x)}(\Omega)}$;
(iv) $|u|_{L^{p(x)}(\Omega)} \to 0$ (*respectively* $\to +\infty$) $\Leftrightarrow \rho(u) \to 0$ (*respectively* $\to +\infty$);
(v) $\rho(u/|u|_{L^{p(x)}(\Omega)}) = 1$.

For every (u, v) and (φ, ψ) in $W := W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$, let

$$\mathcal{F}(u, v) := \int_\Omega F(x, u, v)\,dx.$$

Then,

$$\mathcal{F}'(u, v)(\varphi, \psi) = D_1\mathcal{F}(u, v)(\varphi) + D_2\mathcal{F}(u, v)(\psi),$$

where

$$D_1\mathcal{F}(u, v)(\varphi) = \int_\Omega \frac{\partial F}{\partial u}(x, u, v)\varphi\,dx$$

and

$$D_2\mathcal{F}(u,v)(\psi) = \int_\Omega \frac{\partial F}{\partial v}(x,u,v)\psi\,dx.$$

The Euler–Lagrange functional associated to (1) is given by

$$J(u,v) := \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}\,dx - \mathcal{F}(u,v).$$

It is easy to verify that $J \in C^1(W,\mathbb{R})$ and that

$$J'(u,v)(\varphi,\psi) = D_1J(u,v)(\varphi) + D_2J(u,v)(\psi), \tag{2}$$

where

$$D_1J(u,v)(\varphi) = \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\,dx - D_1\mathcal{F}(u,v)(\varphi),$$

$$D_2J(u,v)(\psi) = \int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\psi\,dx - D_2\mathcal{F}(u,v)(\psi).$$

Let us choose on W the norm $\|\,.\,\|$ defined by

$$\big\|(u,v)\big\| := \max\big\{\|u\|_p + \|v\|_q\big\},$$

where $\|\,.\,\|_p$ (respectively $\|\,.\,\|_q$) is the norm of $W_0^{1,p(x)}(\Omega)$ (respectively $W_0^{1,q(x)}(\Omega)$). The dual space of W will be denoted by W^* and $\|\,.\,\|_*$ will stand for its norm. Therefore

$$\big\|J'(u,v)\big\|_* = \big\|D_1J(u,v)\big\|_{*,p} + \big\|D_2J(u,v)\big\|_{*,q},$$

where $W^{-1,p'(x)}(\Omega)$ (respectively $W^{-1,q'(x)}(\Omega)$) is the dual space of $W_0^{1,p(x)}(\Omega)$ (respectively $W_0^{1,q(x)}(\Omega)$) and $\|\,.\,\|_{*,p}$ (respectively $\|\,.\,\|_{*,q}$) is its norm. Finally, $|\,.\,|_r$ will stand for the norm on $L^{r(x)}(\Omega)$, for every $r \in C_+(\bar{\Omega})$.

Throughout this paper, the letters c, c_i, $i = 1,2,\ldots,$ denote positive constants which may vary from line to line but are independent of the terms which will take part in any limit process.

3. Existence results

Before stating our results, we introduce some natural growth hypotheses on the right-hand side of (1). Theses hypotheses will insure the mountain pass geometry and the Palais–Smale condition for the Euler–Lagrange functional J. We limit ourselves to the subcritical case, i.e., we assume that

(H$_1$) $F(x,s,t) \leqslant c_1 + c_2|s|^{p_1(x)} + c_3|t|^{q_1(x)} + c_4|s|^{\alpha(x)}|t|^{\beta(x)}$, $\forall (x,s,t) \in \Omega \times \mathbb{R}^2$, where $(p_1,q_1,\alpha,\beta) \in C_+(\bar{\Omega})^4$ and

$$p_1 < p^*, \quad q_1 < q^*, \quad \frac{\alpha}{p^*} + \frac{\beta}{q^*} < 1 \quad \text{in } \bar{\Omega},$$
$$p_1^-, \alpha^- > p^+ \quad \text{and} \quad q_1^-, \beta^- > q^+.$$

Precise that under the hypothesis (H_1), the operator $\mathcal{F}' : W \to W^*$ is compact [9]. To guarantee the Palais–Smale condition for the functional J, we will assume moreover the following

(H_2) $\exists M > 0, \exists \theta_1 > p^+, \exists \theta_2 > q^+ : \forall x \in \Omega, \forall (s,t) \in \mathbb{R}^2 : |s|^{\theta_1} + |t|^{\theta_2} \geqslant 2M$, one has

$$0 < F(x,s,t) \leqslant \frac{s}{\theta_1}\frac{\partial F}{\partial s}(x,s,t) + \frac{t}{\theta_2}\frac{\partial F}{\partial t}(x,s,t).$$

Finally, we assume

(H_3) $F(x,s,t) = o(|s|^{p^+} + |t|^{q^+})$ as $(s,t) \to (0,0)$, uniformly w.r.t. $x \in \Omega$.

As we will show later, the hypotheses (H_1) and (H_3) imply the mountain pass geometry for the functional J.

Lemma 2. *Let (u_n, v_n) be a Palais–Smale sequence for the Euler–Lagrange functional J. If (H_2) is satisfied then (u_n, v_n) is bounded.*

Proof. Let (u_n, v_n) be a Palais–Smale sequence for the functional J. This means that $J(u_n, v_n)$ is bounded and $\|J'(u_n, v_n)\|_* \to 0$ as n goes to $+\infty$. Then, there is a positive constant c such that

$$\begin{aligned} c \geqslant J(u_n, v_n) &= \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx + \int_\Omega \frac{1}{q(x)} |\nabla v_n|^{q(x)}\,dx - \int_\Omega F(x,u_n,v_n)\,dx \\ &\geqslant \int_\Omega \left(\frac{1}{p(x)} |\nabla u_n|^{p(x)} - \frac{u_n}{\theta_1}\frac{\partial F}{\partial u}(x,u_n,v_n) \right) dx \\ &\quad + \int_\Omega \left(\frac{1}{q(x)} |\nabla v_n|^{q(x)} - \frac{v_n}{\theta_2}\frac{\partial F}{\partial v}(x,u_n,v_n) \right) dx - c_1, \end{aligned}$$

where c_1 is some positive constant. Then,

$$\begin{aligned} c \geqslant{}& \left(\frac{1}{p^+} - \frac{1}{\theta_1} \right) \int_\Omega |\nabla u_n|^{p(x)}\,dx + \left(\frac{1}{q^+} - \frac{1}{\theta_2} \right) \int_\Omega |\nabla v_n|^{q(x)}\,dx \\ &+ \frac{1}{\theta_1} \int_\Omega \left(|\nabla u_n|^{p(x)} - u_n \frac{\partial F}{\partial u}(x,u_n,v_n) \right) dx \\ &+ \frac{1}{\theta_2} \int_\Omega \left(|\nabla v_n|^{q(x)} - v_n \frac{\partial F}{\partial v}(x,u_n,v_n) \right) dx - c_2 \end{aligned}$$

$$\geqslant \left(\frac{1}{p^+} - \frac{1}{\theta_1}\right) \int_\Omega |\nabla u_n|^{p(x)}\,dx + \left(\frac{1}{q^+} - \frac{1}{\theta_2}\right) \int_\Omega |\nabla v_n|^{q(x)}\,dx$$
$$- \frac{1}{\theta_1} \left\| D_1 J(u_n, v_n) \right\|_{*,p} \|u_n\|_p - \frac{1}{\theta_2} \left\| D_2 J(u_n, v_n) \right\|_{*,q} \|v_n\|_q - c_2.$$

Now, suppose that the sequence (u_n, v_n) is not bounded. Without loss of generality, we may assume $\|u_n\|_p \geqslant \|v_n\|_q$.

Therefore, for n large enough, we get

$$c \geqslant \left(\frac{1}{p^+} - \frac{1}{\theta_1}\right) \|u_n\|_p^{p^-} - \left(\frac{1}{\theta_1} \left\| D_1 J(u_n, v_n) \right\|_{*,p} - \frac{1}{\theta_2} \left\| D_2 J(u_n, v_n) \right\|_{*,q}\right) \|u_n\|_p.$$

But, this cannot hold true since $p^- > 1$. Hence, the sequence (u_n, v_n) is bounded. □

In the following lemma, we show a compactness result. More precisely, we show that every bounded Palais–Smale sequence for the functional J contains a Cauchy subsequence. In the simple case where the functions p and q are constant, the extraction of Cauchy subsequences, based on the inequalities (3), is very standard. Here, we adapt this method to the general case where p and q are in $C_+(\bar{\Omega})$.

Lemma 3. *Let (u_n, v_n) be a bounded Palais–Smale sequence for the Euler–Lagrange functional J. If (H_1) is satisfied then (u_n, v_n) contains a convergent subsequence.*

Proof. Let (u_n, v_n) be a bounded Palais–Smale sequence for the functional J. Then there is a subsequence still denoted by (u_n, v_n) which converges weakly in W. We recall the well-known inequalities

$$\begin{cases} |x-y|^\gamma \leqslant 2^\gamma (|x|^{\gamma-2}x - |y|^{\gamma-2}y) \cdot (x-y) & \text{if } \gamma \geqslant 2, \\ |x-y|^2 \leqslant (|x|+|y|)^{2-\gamma} (|x|^{\gamma-2}x - |y|^{\gamma-2}y) \cdot (x-y) & \text{if } 1 < \gamma < 2, \end{cases} \tag{3}$$

for every x and y in $\mathbb{R}^N$, where $\cdot$ denotes the standard inner product in $\mathbb{R}^N$. We show that (u_n, v_n) contains a Cauchy subsequence. Let us define

$$\mathcal{U}_p = \left\{x \in \Omega:\ p(x) \geqslant 2\right\} \quad \text{and} \quad \mathcal{V}_p = \left\{x \in \Omega:\ 1 < p(x) < 2\right\},$$
$$\mathcal{U}_q = \left\{x \in \Omega:\ q(x) \geqslant 2\right\} \quad \text{and} \quad \mathcal{V}_q = \left\{x \in \Omega:\ 1 < q(x) < 2\right\}.$$

For every $x \in \Omega$, we set

$$\Phi_{n,k} = \left(|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u_k|^{p(x)-2} \nabla u_k\right) \cdot (\nabla u_n - \nabla u_k),$$
$$\Psi_{n,k} = \left(|\nabla u_n| + |\nabla u_k|\right)^{2-p(x)}.$$

Then,

$$\int_{\mathcal{U}_p} |\nabla u_n - \nabla u_k|^{p(x)}\,dx \leqslant c \int_\Omega \Phi_{n,k}\,dx,$$
$$\int_{\mathcal{V}_p} |\nabla u_n - \nabla u_k|^{p(x)}\,dx \leqslant \int_\Omega \Phi_{n,k}^{p(x)/2} \Psi_{n,k}^{(2-p(x))p(x)/2}\,dx.$$

Recall that

$$\begin{aligned}\int_\Omega \Phi_{n,k}\,dx &= \big(D_1 J(u_n,v_n) - D_1 J(u_k,v_k) + D_1\mathcal{F}(u_n,v_n) \\ &\quad - D_1\mathcal{F}(u_k,v_k)\big)(u_n-u_k) \\ &\leqslant \big(\|D_1 J(u_n,v_n)\|_{*,p} + \|D_1 J(u_k,v_k)\|_{*,p}\big)\|u_n-u_k\|_p \\ &\quad + \big(\|D_1\mathcal{F}(u_n,v_n) - D_1\mathcal{F}(u_k,v_k)\|_{*,p}\big)\|u_n-u_k\|_p.\end{aligned}$$

Since $\|J'(u_m,v_m)\|_* \to 0$, as m goes to $+\infty$, and $\mathcal{F}'$ is compact then we can consider that

$$0 \leqslant \int_\Omega \Phi_{n,k}\,dx < 1.$$

If

$$\int_\Omega \Phi_{n,k}\,dx = 0,$$

using the fact that $\Phi_{n,k} \geqslant 0$ in Ω, we obtain that $\Phi_{n,k}=0$.

If

$$0 < \int_\Omega \Phi_{n,k}\,dx < 1, \tag{4}$$

thanks to the Young inequality and (4), we conclude

$$\begin{aligned}&\int_{\mathcal{V}_p} \Phi_{n,k}^{p(x)/2}\left(\int_{\mathcal{V}_p}\Phi_{n,k}(y)\,dy\right)^{-1/2}\Psi_{n,k}^{(2-p(x))p(x)/2}\,dx \\ &\quad\leqslant \int_{\mathcal{V}_p}\left(\Phi_{n,k}\left(\int_{\mathcal{V}_p}\Phi_{n,k}(y)\,dy\right)^{-1/p(x)} + \Psi_{n,k}^{p(x)}\right)dx \\ &\quad\leqslant 1 + \int_\Omega \Psi_{n,k}^{p(x)}\,dx.\end{aligned}$$

Hence,

$$\int_{\mathcal{V}_p} |\nabla u_n - \nabla u_k|^{p(x)}\,dx \leqslant \left(\int_\Omega \Phi_{n,k}\,dx\right)^{1/2}\left(1+\int_\Omega \Psi_{n,k}^{p(x)}\,dx\right).$$

Notice that $\int_\Omega \Psi_{n,k}^{p(x)}\,dx$ is bounded uniformly w.r.t. n and k. Using again the fact that $\|J'(u_m,v_m)\|_* \to 0$, as m tends to $+\infty$, and $\mathcal{F}'$ is compact, we obtain, up to a subsequence,

$$\lim_{n,k\to+\infty}\int_\Omega \Phi_{n,k}\,dx = 0.$$

Therefore, passing if necessary to a subsequence, we have

$$\lim_{n,k\to+\infty}\int_\Omega |\nabla u_n - \nabla u_k|^{p(x)}\,dx = 0.$$

Applying the same arguments, we find a subsequence of (u_n, v_n) such that

$$\lim_{n,k\to+\infty}\int_\Omega |\nabla v_n - \nabla v_k|^{q(x)}\,dx = 0.$$

Hence, up to a subsequence, one has

$$\lim_{n,k\to+\infty}\big\|(u_n, v_n) - (u_k, v_k)\big\| = 0,$$

which implies that (u_n, v_n) contains a Cauchy subsequence and therefore contains a strongly convergent subsequence. This ends the proof. □

Remark 4. The proof of the previous lemma can be showed also via the Lebesgue dominated convergence theorem.

Now, we can state the following existence result.

Theorem 5. *If the hypotheses* (H_1)–(H_3) *hold true, then the problem* (1) *has at least one weak solution.*

Proof. First, let λ, μ in $C_+(\bar{\Omega})$ satisfying $\lambda(x) < \mu^*(x)$ in Ω. Using the continuous embedding $W_0^{1,\mu(x)}(\Omega) \subset L^{\lambda(x)}(\Omega)$ and Lemma 1, we get $\exists\delta_1, \delta_2$: $0 < \delta_1, \delta_2 < 1$ such that

$$\forall f \in W_0^{1,\mu(x)}(\Omega), \quad \|f\|_{W_0^{1,\mu(x)}(\Omega)} \leqslant \delta_1 \quad \Rightarrow \quad \|f\|_{L^{\lambda(x)}(\Omega)} \leqslant \delta_2. \tag{5}$$

On the other hand, let $\tilde{\alpha}$, $\tilde{\beta}$ be two continuous and positive functions on $\bar{\Omega}$ such that

$$\frac{\alpha(x)+\tilde{\alpha}(x)}{p^*(x)} + \frac{\beta(x)+\tilde{\beta}(x)}{q^*(x)} = 1, \quad \forall x \in \bar{\Omega}.$$

For $\|(u, v)\| \ll 1$, using the Young inequality, Lemma 1 and (5), we obtain

$$\begin{aligned}\int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}\,dx &\leqslant \left(\int_\Omega |u|^{\alpha(x)}\,dx\right)^{(p^*/(\alpha+\tilde{\alpha}))^-} + \left(\int_\Omega |v|^{\beta(x)}\,dx\right)^{(q^*/(\beta+\tilde{\beta}))^-}\\ &\leqslant \int_\Omega |u|^{\alpha(x)}\,dx + \int_\Omega |v|^{\beta(x)}\,dx \leqslant c\big(\|u\|_p^{\alpha^-} + \|v\|_q^{\beta^-}\big).\end{aligned}$$

Finally, notice that under the hypothesis (H_1), we have the continuous embeddings $W_0^{1,p(x)}(\Omega) \subset L^{p^+}(\Omega)$ and $W_0^{1,q(x)}(\Omega) \subset L^{q^+}(\Omega)$. That is, $\exists C_1 > 0$, $\exists C_2 > 0$ such that

$$\|u\|_{L^{p^+}(\Omega)} \leqslant C_1\|u\|_{W_0^{1,p(x)}(\Omega)} \quad \text{and} \quad \|v\|_{L^{q^+}(\Omega)} \leqslant C_2\|v\|_{W_0^{1,q(x)}(\Omega)}.$$

Now, let $\varepsilon > 0$ such that

$$\varepsilon \max\bigl(C_1^{p^+}, C_2^{q^+}\bigr) \leqslant \frac{1}{2}\min(p^+, q^+).$$

It follows from (H_1) and (H_3) that $\forall (x,s,t) \in \Omega \times \mathbb{R}^2$,

$$F(x,s,t) \leqslant \varepsilon\bigl(|s|^{p^+} + |t|^{q^+}\bigr) + C(\varepsilon)\bigl(|s|^{p_1(x)} + |t|^{q_1(x)} + |s|^{\alpha(x)}|t|^{\beta(x)}\bigr).$$

Therefore, for $\|(u,v)\|$ sufficiently small, we get

$$\begin{aligned} J(u,v) &\geqslant \frac{1}{p^+}\|u\|_p^{p^+} - \varepsilon C_1^{p^+}\|u\|_p^{p^+} + \frac{1}{q^+}\|v\|_q^{q^+} - \varepsilon C_2^{q^+}\|v\|_q^{q^+} \\ &\quad - C(\varepsilon)\int_\Omega \bigl(|u|^{p_1(x)} + |v|^{q_1(x)} + |u|^{\alpha(x)}|v|^{\beta(x)}\bigr)\,dx \\ &\geqslant \frac{\|u\|_p^{p^+}}{2p^+} + \frac{\|v\|_q^{q^+}}{2q^+} - C(\varepsilon)\bigl(\|u\|_p^{p_1^-} + \|v\|_q^{q_1^-} + c\|u\|_p^{\alpha^-} + c\|v\|_q^{\beta^-}\bigr). \end{aligned}$$

Since $p_1^-, \alpha^- > p^+$ and $q_1^-, \beta^- > q^+$, there is $r > 0$ and $c' > 0$ such that $J(u,v) \geqslant c'$ for every $(u,v) \in W$ satisfying $\|(u,v)\| = r$.

On the other hand, we claim that the assumption (H_2) implies the following assertion: for every $x \in \bar{\Omega}$, $s,t \in \mathbb{R}$, the inequality

$$F(x,s,t) \geqslant c\bigl(\bigl(|s|^{\theta_1} + |t|^{\theta_2}\bigr) - 1\bigr)$$

holds true. Indeed, consider the compact subset K of $\mathbb{R}^2$ defined by

$$K := \left\{(s,t) \in \mathbb{R}^2\colon\ \frac{|s|^{\theta_1} + |t|^{\theta_2}}{2} = 1\right\}.$$

For every $(s,t) \in K$, we introduce the function

$$G(x,\tau) := F\bigl(x, |\tau|^{1/\theta_1}s, |\tau|^{1/\theta_2}t\bigr)$$

defined on $\bar{\Omega} \times \mathbb{R}$. Then,

$$\tau\frac{\partial G}{\partial \tau} = \frac{s}{\theta_1}|\tau|^{1/\theta_1}\partial_2 F\bigl(x, |\tau|^{1/\theta_1}s, |\tau|^{1/\theta_2}t\bigr) + \frac{t}{\theta_2}|\tau|^{1/\theta_2}\partial_3 F\bigl(x, |\tau|^{1/\theta_1}s, |\tau|^{1/\theta_2}t\bigr),$$

where ∂_i denotes the partial derivative of F w.r.t. its ith variable. Therefore, for $|\tau| \geqslant M$, we get

$$\tau\frac{\partial G}{\partial \tau} \geqslant G(x,\tau) > 0$$

which implies that $G(x,\tau) \geqslant \frac{G(x,M)}{M}|\tau|$. This means that

$$F\bigl(x, |\tau|^{1/\theta_1}s, |\tau|^{1/\theta_2}t\bigr) \geqslant \frac{F(x, M^{1/\theta_1}s, M^{1/\theta_2}t)}{M}|\tau|, \quad \forall |\tau| \geqslant M.$$

Set

$$c_1 := \min_{x\in\bar{\Omega}}\min_{(s,t)\in K}\frac{F(x, M^{1/\theta_1}s, M^{1/\theta_2}t)}{M} > 0.$$

Then $F(x, |\tau|^{1/\theta_1}s, |\tau|^{1/\theta_2}t) \geqslant c_1|\tau| - c_2, \forall \tau \in \mathbb{R}$. Moreover, every $(u, v) \in \mathbb{R}^2$ can be rewritten as

$$(u, v) = \left(\left(|u|^{\theta_1} + |v|^{\theta_2}\right)^{1/\theta_1} u', \left(|u|^{\theta_1} + |v|^{\theta_2}\right)^{1/\theta_2} v'\right),$$

where $(u', v') \in K$. Therefore, there is $c > 0$ such that $F(x, s', t') \geqslant c(|s'|^{\theta_1} + |t'|^{\theta_2} - 1)$, $\forall (s', t') \in \mathbb{R}^2$. This achieves our claim.

Now, let $\tilde{u} \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$, $\tilde{v} \in W_0^{1,q(x)}(\Omega) \setminus \{0\}$ and $t > 1$. Then,

$$\begin{aligned} J(t\tilde{u}, t\tilde{v}) &= \int_\Omega \frac{t^{p(x)}|\nabla \tilde{u}|^{p(x)}}{p(x)}\,dx + \int_\Omega \frac{t^{q(x)}|\nabla \tilde{v}|^{q(x)}}{q(x)}\,dx - \int_\Omega F(x, t\tilde{u}, t\tilde{v})\,dx \\ &\leqslant t^{p^+} \int_\Omega \frac{|\nabla \tilde{u}|^{p(x)}}{p(x)}\,dx + t^{q^+} \int_\Omega \frac{|\nabla \tilde{v}|^{q(x)}}{q(x)}\,dx - ct^{\theta_1} \int_\Omega |\tilde{u}|^{\theta_1}\,dx \\ &\quad - ct^{\theta_2} \int_\Omega |\tilde{v}|^{\theta_2}\,dx - c|\Omega|. \end{aligned}$$

Therefore,

$$\lim_{t \to +\infty} J(t\tilde{u}, t\tilde{v}) = -\infty$$

which implies that J possesses the mountain pass geometry. Applying Lemmas 2 and 3, we conclude that J has at least one nontrivial critical point [13,17,18]. This achieves the proof. □

4. Infinitely many solutions

In this section, we prove under some symmetry condition on the function F that the problem (1) possesses infinitely many nontrivial weak solutions. The proof is based on Bartsch's fountain theorem [1]. Since $W_0^{1,p(x)}$ and $W_0^{1,q(x)}$ are reflexive and separable (and their dual), then W and W^* are too. Let $(e_i)_{i\in\mathbb{N}} \subset W$ and $(e_i^*)_{i\in\mathbb{N}} \subset W^*$ such that

$$W = \overline{\operatorname{span}\{e_i\colon\ i \in \mathbb{N}\}}, \qquad W^* = \overline{\operatorname{span}\{e_i^*\colon\ i \in \mathbb{N}\}}$$

and

$$\left\langle e_i^*, e_j \right\rangle = \delta_{ij}, \quad \forall i, j \in \mathbb{N},$$

where $\langle .\,, .\rangle$ denotes the duality product between W^* and W, and δ_{ij} is the Kronecker symbol. We define $X_j := \mathbb{R}e_j$, $Y_k := \bigoplus_{j=0}^{k} X_j$ and $Z_k := \bigoplus_{j=k}^{\infty} X_j$.

Let us recall the version of the Fountain theorem which will be used in the sequel.

Theorem 6. *Let $J \in C^1(W, \mathbb{R})$ be an even functional, where $(W, \|\,.\,\|)$ is a separable and reflexive Banach space. If for every $k \in \mathbb{N}$, there are $0 < r_k < \rho_k$ such that*

(i) $\max\{J(u)\colon\ u \in Y_k,\ \|u\| = \rho_k\} \leqslant 0$,

(ii) $\inf\{J(u)\colon u \in Z_k,\ \|u\| = r_k\} \to +\infty$ *as* $k \to +\infty$,
(iii) J *satisfies the Palais–Smale condition for every level,*

then J *has an unbounded sequence of critical points.*

For every $a > 1$, $u \in L^a(\Omega)$ and $v \in L^a(\Omega)$, we define

$$\big|(u, v)\big|_a := \max\big\{|u|_a, |v|_a\big\}.$$

In (H_1), let $\tilde{\alpha}, \tilde{\beta}$ be two continuous and positive functions on $\bar{\Omega}$ such that

$$\frac{\alpha(x) + \tilde{\alpha}(x)}{p^*(x)} + \frac{\beta(x) + \tilde{\beta}(x)}{q^*(x)} = 1, \quad \forall x \in \bar{\Omega}.$$

Set

$$a := \max_{x \in \bar{\Omega}} \left\{ \frac{\alpha(x) + \tilde{\alpha}(x)}{p^*(x)}, \frac{\beta(x) + \tilde{\beta}(x)}{q^*(x)}, p_1(x), q_1(x) \right\},$$
$$b := \min_{x \in \bar{\Omega}} \left\{ \frac{\alpha(x) + \tilde{\alpha}(x)}{p^*(x)}, \frac{\beta(x) + \tilde{\beta}(x)}{q^*(x)}, p_1(x), q_1(x) \right\}.$$

Then we have the following

Lemma 7. *Define*

$$\beta_k := \sup\big\{\big|(u, v)\big|_a\colon \ \big\|(u, v)\big\| = 1,\ (u, v) \in Z_k\big\}.$$

Then, $\lim_{k \to +\infty} \beta_k = 0$.

Proof. It is clear that the sequence (β_k) is nonincreasing and positive. Let $\ell \geqslant 0$ such that $\lim_{k \to +\infty} \beta_k = \ell$, and $(u_k, v_k) \in Z_k$ with $\|(u_k, v_k)\| = 1$, $0 \leqslant \ell - |(u_k, v_k)|_a \leqslant 1/k$. Passing if necessary to a subsequence, there is $(u, v) \in W$ such that (u_k, v_k) converges weakly to (u, v) in W. On the other hand, for every $j \in \mathbb{N}$,

$$\big\langle e_j^*, (u, v)\big\rangle = \lim_{k \to +\infty} \big\langle e_j^*, (u_k, v_k)\big\rangle = 0.$$

Therefore, $(u, v) = 0$ and u_k (respectively v_k) converges weakly to u (respectively v) in $W_0^{1,p(x)}(\Omega)$ (respectively $W_0^{1,q(x)}(\Omega)$). By virtue of the compact embeddings $W_0^{1,p(x)}(\Omega) \subset L^a(\Omega)$, $W_0^{1,q(x)}(\Omega) \subset L^a(\Omega)$, it follows that (u_k, v_k) converges strongly to 0 in W and finally that $\ell = 0$. $\square$

In the following theorem, we show the existence of infinitely many solutions to the problem (1) under some symmetry assumption on F. We confine ourselves to the case where $p(x) = q(x)$ for every $x \in \bar{\Omega}$. Notice that the result remains valid for $p \neq q$ by adding some slight (and technical) changes in the hypothesis (H_2).

Theorem 8. *If* $F(x, u, v)$ *is even in* u, v *and satisfies the hypotheses* (H_1) *and* (H_2), *then* (1) *possesses infinitely many* (*pairs*) *of solutions with unbounded energy.*

Proof. It suffices to show that J has an unbounded sequence of critical points. The proof is based on the fountain theorem (see [1]). Let $(u_k, v_k) \in Z_k$ such that $\|(u_k, v_k)\| = r_k \geqslant 1$ (r_k will be specified below). In what follows, we will use the mean value theorem in the following form: For every $\gamma \in C_+(\bar{\Omega})$, $u \in L^\gamma(\Omega)$, there is $\xi \in \Omega$ such that

$$\int_\Omega |u|^{\gamma(x)}\,dx = |u|_\gamma^{\gamma(\xi)}.$$

Indeed, it is well known that there is $\xi \in \Omega$ such that

$$1 = \int_\Omega \bigl(|u|/|u|_\gamma\bigr)^{\gamma(x)}\,dx = \int_\Omega |u|^{\gamma(x)}\,dx/|u|_\gamma^{\gamma(\xi)},$$

and the claim is complete. It follows,

$$\begin{aligned}
J(u_k, v_k) &\geqslant \frac{1}{p^+}\|u_k\|_p^{p^-} + \frac{1}{q^+}\|v_k\|_q^{q^-} \\
&\quad - c\int_\Omega \bigl(1 + |u_k|^{p_1(x)} + |v_k|^{q_1(x)} + |u_k|^{\alpha(x)}|v_k|^{\beta(x)}\bigr)\,dx \\
&\geqslant \frac{1}{p^+}\|u_k\|_p^{p^-} + \frac{1}{q^+}\|v_k\|_q^{q^-} - c|u_k|_{p_1}^{p_1(\xi_1^k)} - c|v_k|_{q_1}^{q_1(\xi_2^k)} \\
&\quad - c|u_k|_{p^*\alpha/(\alpha+\tilde{\alpha})}^{p^*\alpha/(\alpha+\tilde{\alpha})(\eta_1^k)} - c|v_k|_{p^*\beta/(\beta+\tilde{\beta})}^{p^*\beta/(\beta+\tilde{\beta})(\eta_2^k)} - c|\Omega|,
\end{aligned}$$

where $\xi_1^k, \xi_2^k, \eta_1^k, \eta_2^k \in \Omega$ and $|\Omega|$ denotes the measure of Ω. Therefore,

$$\begin{aligned}
J(u_k, v_k) &\geqslant \frac{1}{p^+}\|u_k\|_p^{p^-} + \frac{1}{q^+}\|v_k\|_q^{q^-} - c_1|u_k|_a^{p_1(\xi_1^k)} - c_1|v_k|_a^{q_1(\xi_2^k)} \\
&\quad - c_1|u_k|_a^{p^*\alpha/(\alpha+\tilde{\alpha})(\eta_1^k)} - c_1|v_k|_a^{p^*\beta/(\beta+\tilde{\beta})(\eta_2^k)} - c|\Omega| \\
&\geqslant \frac{1}{\max(p^+, q^+)}\bigl\|(u_k, v_k)\bigr\|^{\min(p^-,q^-)} - c_1\bigl(\beta_k\bigl\|(u_k, v_k)\bigr\|\bigr)^{p_1(\xi_1^k)} \\
&\quad - c_1\bigl(\beta_k\bigl\|(u_k, v_k)\bigr\|\bigr)^{q_1(\xi_2^k)} - c_1\bigl(\beta_k\bigl\|(u_k, v_k)\bigr\|\bigr)^{p^*\alpha/(\alpha+\tilde{\alpha})(\eta_1^k)} \\
&\quad - c_1\bigl(\beta_k\bigl\|(u_k, v_k)\bigr\|\bigr)^{p^*\beta/(\beta+\tilde{\beta})(\eta_2^k)} - c|\Omega| \\
&\geqslant \frac{1}{\max(p^+, q^+)}\bigl\|(u_k, v_k)\bigr\|^{\min(p^-,q^-)} - c_2\beta_k^b\bigl\|(u_k, v_k)\bigr\|^a - c|\Omega|,
\end{aligned}$$

where a, b are defined above. At this stage, we fix r_k as follows:

$$r_k := \left(\frac{\beta_k^{-b}}{2c_2\max(p^+, q^+)}\right)^{1/(a-\min(p^-,q^-))} \to +\infty \quad \text{as } k \to +\infty.$$

Consequently, if $\|(u_k, v_k)\| = r_k$ then

$$J(u_k, v_k) \geqslant \frac{1}{2\max(p^+, q^+)}\bigl\|(u_k, v_k)\bigr\|^{\min(p^-,q^-)} - c|\Omega|.$$

On the other hand, it is known from (H_2) that $F(x,u,v) \geqslant c(|u|^{\theta_1} + |v|^{\theta_2} - 1)$, for every $x \in \Omega$ and $u, v \in \mathbb{R}$. Whence, if $(u,v) \in Y_k$ with $u \neq 0$, $v \neq 0$, we get from above that

$$\lim_{t \to +\infty} J(tu, tv) = -\infty.$$

This implies that

$$\max\big\{J(u,v):\ \|(u,v)\| = \rho_k,\ (u,v) \in Y_k\big\} \leqslant 0$$

for every ρ_k large enough. Applying the fountain theorem, we achieve the proof. □

Acknowledgment

I am very grateful to Prof. Mokhtar Kirane and Prof. Xian-Ling Fan who provided me interesting references on this subject.

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